

# A New Superintegrable Hamiltonian

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## Abstract

We identify a new superintegrable Hamiltonian in 3 degrees of freedom, obtained as a reduction of pure Keplerian motion in 6 dimensions. The new Hamiltonian is a generalization of the Keplerian one, and has the familiar  $1/r$  potential with three barrier terms preventing the particle crossing the principal planes. In 3 degrees of freedom, there are 5 functionally independent integrals of motion, and all bound, classical trajectories are closed and strictly periodic. The generalisation of the Laplace-Runge-Lenz vector is identified and shown to provide functionally independent isolating integrals. They are quartic in the momenta and do not arise from separability of the Hamilton-Jacobi equation. A formulation of the system in action-angle variables is presented.

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## I. INTRODUCTION

The Kepler problem is well known to be superintegrable – that is, it has five functionally independent integrals of motion. They are the energy and the components of angular momentum and the Laplace-Runge-Lenz vectors, obtainable by separating the Hamilton-Jacobi equation in spherical polar and rotational parabolic coordinates [1]. For Hamiltonians with three degrees of freedom, the existence of five integrals of motion implies that every bound trajectory is closed.

Sommerfeld and Born, in the days of the old quantum theory, appear to have been the first to realize that if a potential is separable in more than one coordinate system, it possesses additional isolating functionally independent integrals [2, 3]. The first systematic inquiry into this problem was begun by Winternitz, Smorodinsky and co-workers, who found every potential in two degrees of freedom for which the Hamilton-Jacobi equation is separable in more than one way [4]. Subsequently, they extended this work to three degrees of freedom by finding every potential separable in spherical polars and at least one additional coordinate system [5]. Evans [6] then completed this work by investigating all the remaining possibilities. A useful introduction to the subject of superintegrability, as well as summary of recent work, is given in the conference proceedings of Tempesta et al. [7].

If the Hamilton-Jacobi equation separates, then the corresponding integral of motion is necessarily linear or quadratic in the canonical momenta. Consequently, all the superintegrable systems listed in [4, 5, 6] have integrals that are quadratic in the momenta. As an example, let us consider the Keplerian Hamiltonian

$$H = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{k}{r}, \quad (1)$$

where  $k$  is a real positive constants. Letting  $\mathbf{r}$  denote the position vector, then the integrals are the energy  $E$ , the components of the angular momentum vector  $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}}$  and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \dot{\mathbf{r}} \times \mathbf{L} - \frac{k}{r}\mathbf{r}, \quad (2)$$

all of which are at most quadratic in the velocities.

Superintegrable systems with higher-order integrals are known [8], although they are extremely scarce. Examples include the anisotropic harmonic with rational frequency ratio

$\ell : m : n$  where  $\ell + m + n \geq 5$ , i.e.,

$$H = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \ell^2 x^2 + m^2 y^2 + n^2 z^2. \quad (3)$$

The potential separates in rectangular cartesian and possesses two commuting quadratic integrals. There are two additional integrals which may be taken as polynomials of degree  $\ell + m - 1$  and  $\ell + n - 1$  [9]. The Calogero potential in a harmonic well

$$H = \frac{1}{2}|\dot{\mathbf{r}}|^2 + k(x^2 + y^2 + z^2) + \frac{k_1}{(x - y)^2} + \frac{k_1}{(y - z)^2} + \frac{k_1}{(z - x)^2}, \quad (4)$$

is known to be super-integrable and possesses an integral of the motion that is cubic in the velocities [10, 11].

In this paper, we introduce a new superintegrable Hamiltonian, namely

$$H = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{k}{r} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}. \quad (5)$$

This is recognized as a generalization of the familiar Keplerian Hamiltonian. The constants  $k_1, k_2$  and  $k_3$  are taken as positive, so the Hamiltonian is perfectly physical and motion is confined to, say, the octant  $x > 0, y > 0$  and  $z > 0$ . The terms involving the  $k_i$  correspond to repulsive barriers preventing the orbit crossing any of the principal planes. In Section II, we present numerical integrations showing that the orbits give closed curves in all cases. This motivates a search for the isolating integrals, one of which is found to be quartic in the momenta in Section III. Finally, an action angle formalism is given in Section IV and the relation to the Kepler problem discussed.

## II. EVIDENCE OF SUPERINTEGRABILITY

### A. Analytic Proof

Let us recall that Keplerian motion in  $N$  degrees of freedom always possesses  $2N - 1$  functionally independent integrals of motion [12]. Specialising to 6 degrees of freedom, we have the Hamiltonian

$$H = \frac{1}{2}|\mathbf{p}|^2 - \frac{k}{|\mathbf{s}|} \quad (6)$$

where  $s$  has Cartesian coordinates  $(s_1, s_2, s_3, s_4, s_5, s_6)$ . Now, let us introduce coordinates  $(x, y, z, \theta_x, \theta_y, \theta_z)$  according to

$$\begin{aligned} s_1 &= x \cos \theta_x, & s_2 &= x \sin \theta_x \\ s_3 &= y \cos \theta_y, & s_4 &= y \sin \theta_y \\ s_5 &= z \cos \theta_z, & s_6 &= z \sin \theta_z \end{aligned}$$

The Hamiltonian becomes

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + \frac{p_{\theta_1}^2}{x^2} + \frac{p_{\theta_2}^2}{y^2} + \frac{p_{\theta_3}^2}{z^2}) - \frac{k}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$$

The coordinates  $(\theta_1, \theta_2, \theta_3)$  are ignorable, so we obtain a new Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{k}{r} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \quad (7)$$

where  $k_1, k_2$  and  $k_3$  are the constant values of the momenta conjugate to the ignorable coordinates. In the original 6 degrees of freedom Hamiltonian (6), every bound trajectory is closed. Consequently, in the reduced 3 degrees of freedom Hamiltonian (7), every bound trajectory is also closed. Evidently, the proof can be readily generalised to  $N$  degrees of freedom.

## B. A Sampler of Orbits

It is interesting to investigate characteristic orbits corresponding to the Hamiltonian (7). Using a standard Burlisch-Stoer code [13], the motion of a unit mass particle was followed in the octant with  $x > 0$ ,  $y > 0$  and  $z > 0$ . Stepsize and tolerances were set to maintain accuracy to a level of around  $10^{-12}$  relative energy change, and integration lengths were typically tens of periods.

Starting with initial conditions that would give a circular orbit in a true Keplerian potential, the effect of relatively weak barriers can be seen in Figure 1. The orbit appears similar in shape to the Keplerian ellipse but reflects off the three axes planes. In this case, as a consequence of the  $k_i$  being equal and the symmetry in the initial phase space position, the orbit is confined to a plane. The effect of larger barriers can be seen through increasing the  $k_i$  by a factor of ten, as shown in Figure 2. The orbit is now further distorted, and takes the form of a figure-of-eight. If the  $k_i$  are not equal, the orbit is still closed, as shown in

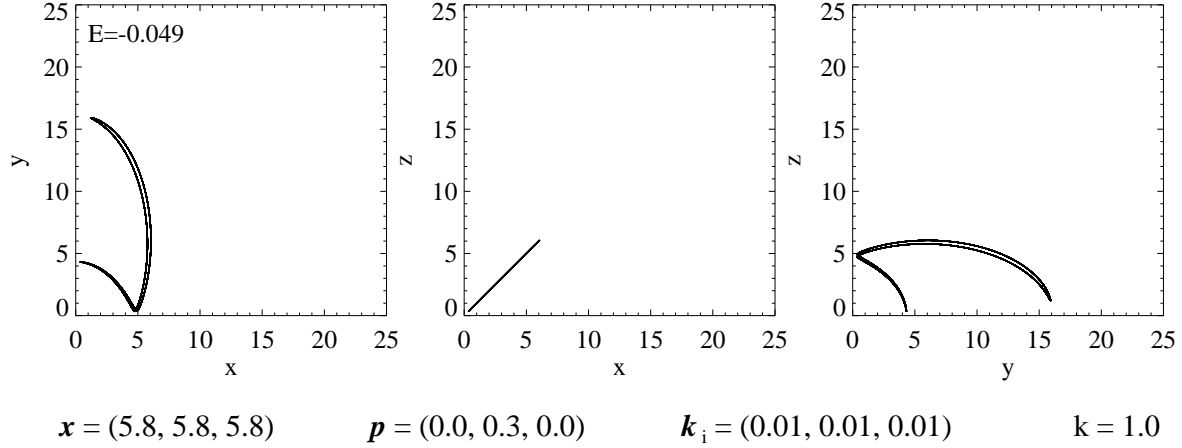


FIG. 1: A perturbed circular Keplerian orbit in the potential. The three panels show the projections in the  $x - y$ ,  $x - z$  and  $y - z$  planes respectively from left to right. The initial conditions are given (to 2 s.f.) below the plot and the energy in the top corner. Note that although in this case the orbit lies in a plane this is not generally true.

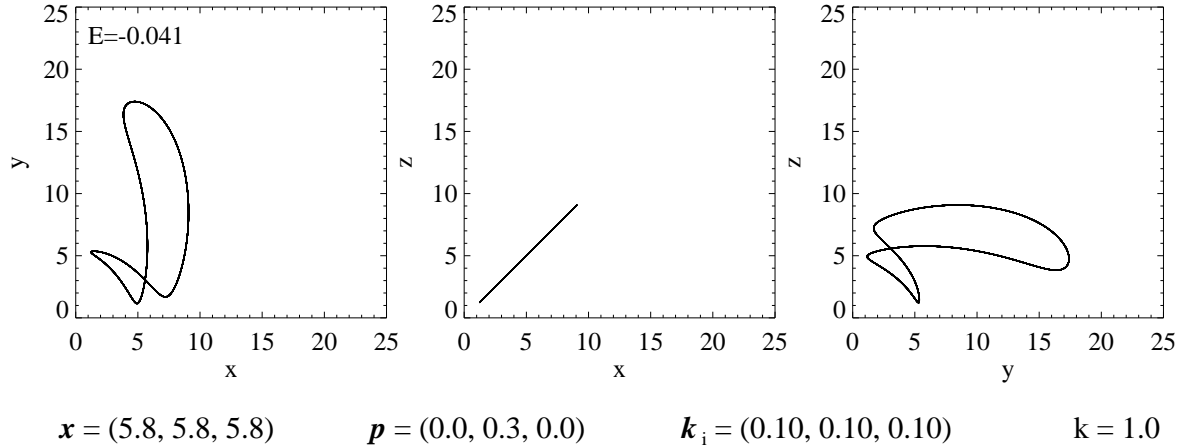


FIG. 2: As for Figure 1 but now the centrifugal barriers are an order of magnitude larger.

Figures 3 and 4. The latter of these two cases has a different set of initial conditions, which would place it on an initially elliptical orbit in the true Keplerian problem.

Many more initial conditions and combinations of parameter values were investigated. In all bound cases, every orbit is closed and strictly periodic, which is reassuring confirmation of the existence of a fifth isolating integral.

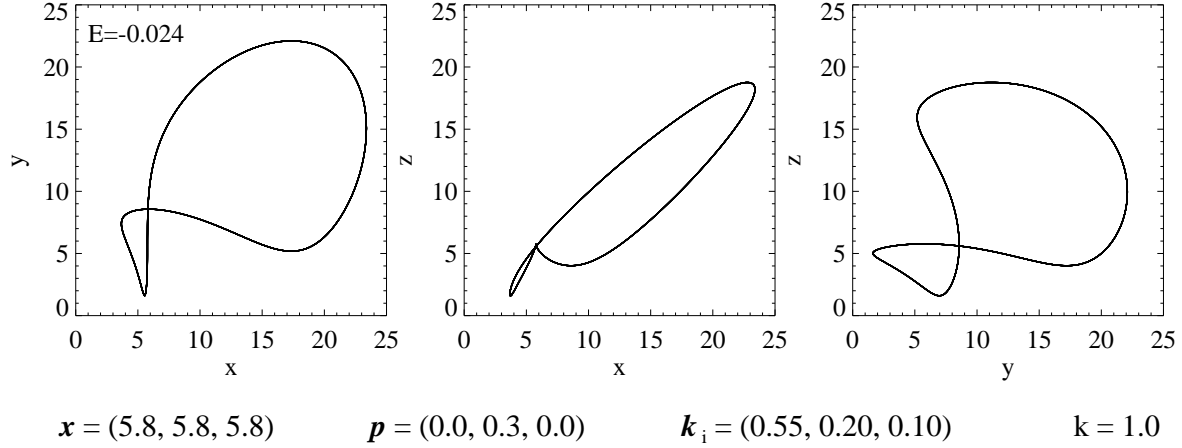


FIG. 3: As for Figure 2 but now the centrifugal barriers are different and the motion is no longer confined to a plane.

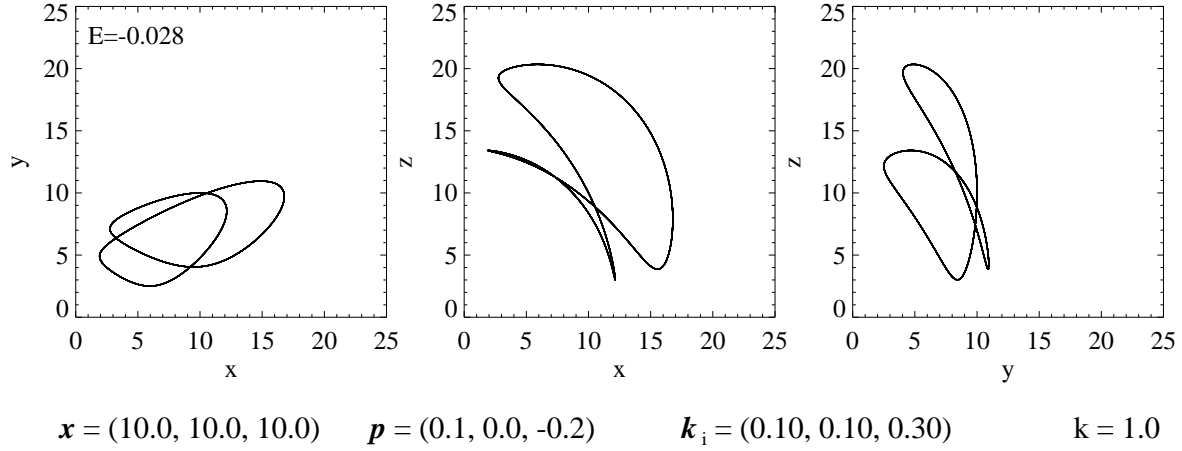


FIG. 4: This orbit in the potential has initial conditions that would place it on an elliptic orbit in the original Kepler problem.

### III. THE INTEGRALS OF MOTION

Makarov et al. [5] and Evans [6] already showed that all Hamiltonians of the form

$$H = \frac{1}{2}|\dot{\mathbf{r}}|^2 + F(r) + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}. \quad (8)$$

possess four isolating integrals of motion, arising from separability of the Hamilton-Jacobi equation in the spherical polar and conical coordinate systems. Here,  $F(r)$  is an arbitrary function of the spherical polar radius. The four isolating integrals are the energy  $E$  and

three generalizations of the angular momentum components, namely

$$E = \frac{1}{2}|\mathbf{p}|^2 + F(r) + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \quad (9)$$

$$I_1 = \frac{1}{2}L_1^2 + \frac{k_2 z^2}{y^2} + \frac{k_3 y^2}{z^2} \quad (10)$$

$$I_2 = \frac{1}{2}L_2^2 + \frac{k_1 z^2}{x^2} + \frac{k_3 x^2}{z^2} \quad (11)$$

$$I_3 = \frac{1}{2}L_3^2 + \frac{k_1 y^2}{x^2} + \frac{k_2 x^2}{y^2} \quad (12)$$

where  $\mathbf{p}$  and  $\mathbf{L}$  are the linear momentum and angular momentum vectors. As Eq (5) is of this form, four of the integrals of motion are already known and arise from separability. The puzzle is that there is a fifth integral whose form is unknown and which does not arise from separability.

In the case where one of the barriers, say that in the  $x = y = 0$  plane, vanishes, the fifth integral is known to be [6]

$$I_4 = L_1 p_2 - p_1 L_2 - 2z \left( -\frac{k}{2r} + \frac{k_1}{x^2} + \frac{k_2}{y^2} \right) \quad (13)$$

and follows from separability in the rotational parabolic coordinate system. If the fifth integral for the general problem (5) is quartic, it must reduce to the above integral in the limit  $k_3 \rightarrow 0$ . This suggests taking the ansatz

$$I_4 = \left( L_1 p_2 - p_1 L_2 - 2z \left( -\frac{k}{2r} + \frac{k_1}{x^2} + \frac{k_2}{y^2} \right) \right)^2 + k_3 g(\mathbf{x}, \mathbf{p}) \quad (14)$$

where  $g(\mathbf{x}, \mathbf{p})$  is a function of both position and momentum yet to be determined. Requiring the Poisson bracket of  $I_4$  with the Hamiltonian to vanish leads to a solution for  $g$ . So, we arrive at an isolating integral of the form

$$I_4 = \left( (\mathbf{L} \times \mathbf{p})_3 - 2z \left( -\frac{k}{2r} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \right) \right)^2 + \frac{2k_3}{z^2} (\mathbf{r} \cdot \mathbf{p})^2 \quad (15)$$

It is not yet proven that this is a functionally independent integral, as it is possible to construct an infinite number of quartic integrals from combinations of the four existing quadratic integrals. To test for functional independence, the  $5 \times 6$  Jacobian

$$\frac{\partial(E, I_1, I_2, I_3, I_4)}{\partial(x_i, p_i)} \quad (16)$$

can be constructed and shown to be of rank 5. Thus, the integral given in Eq (15) is the fifth functionally independent isolating integral of motion for the Hamiltonian.

In the case when the  $k_i$  are all zero, this integral reduces to the  $z$ -component of the Laplace-Runge-Lenz vector. In fact, if we work through the same derivation but make the cyclic permutations  $x \rightarrow y \rightarrow z$ , two more integrals are obtained, They are the equivalents of Eq (15) with the coordinates permuted, and reduce to the  $x$ - and  $y$ -components of the Laplace-Runge-Lenz vector. As is expected they are not functionally independent, and this is easily demonstrated by including them in the Jacobian and noting that it remains of rank 5. The three components are related through

$$I_{4x} + I_{4y} + I_{4z} = 4E(I_1 + I_2 + I_3 + k_1 + k_2 + k_3) + k^2 \quad (17)$$

where  $I_{4z}$  is the integral given by Eq (15) and  $I_{4x}$  and  $I_{4y}$  the cyclicly permuted versions.

#### IV. ACTION-ANGLE VARIABLES

It is also possible to solve this general problem in action-angle variables. This is worthwhile as it provides insight into the relationship of our new superintegrable Hamiltonian with the Kepler problem. Following [14], the actions can be shown to be

$$\begin{aligned} J_\phi &= \oint p_\phi d\phi = 2\sqrt{2}\pi \left( \sqrt{I_2} - \sqrt{k_1} - \sqrt{k_2} \right) \\ J_\theta &= \oint p_\theta d\theta = 2\pi \left( \sqrt{2I_1} - \sqrt{2I_2} - \sqrt{2k_3} \right) \\ J_r &= \oint p_r dr = 2\pi \left( -\sqrt{2I_1} - \frac{k}{\sqrt{-2E}} \right) \end{aligned} \quad (18)$$

and hence

$$E = \frac{-2k^2\pi^2}{J_r + J_\theta + J_\phi + 2\sqrt{2}\pi(\sqrt{k_1} + \sqrt{k_2} + \sqrt{k_3})^2} \quad (19)$$

As expected, the Hamiltonian depends on the actions only through the combination  $J_r + J_\theta + J_\phi$ , implying that the three frequencies of the classical motion are the same. If  $(w_r, w_\theta, w_\phi)$  are the angles conjugate to (19), then we can make a canonical transformation to new action-angle coordinates  $(J_1, J_2, J_3, w_1, w_2, w_3)$ , using the generating function

$$F = (w_\phi - w_\theta)J_1 + (w_\theta - w_r)J_2 + w_r J_3 \quad (20)$$

The new actions are related to the old via

$$J_1 = J_\phi, \quad J_2 = J_\theta + J_\phi, \quad J_3 = J_r + J_\theta + J_\phi, \quad (21)$$



and so the Hamiltonian becomes

$$H = E = \frac{-2k^2\pi^2}{J_3 + 2\sqrt{2}\pi(\sqrt{k_1} + \sqrt{k_2} + \sqrt{k_3})^2} \quad (22)$$

and depends on only one of the new actions. Using Hamilton's equations, we see that the angle  $w_3$  increases linearly with time, whilst the angles  $w_1$  and  $w_2$  are the additional integrals of motion. They can be found by explicit construction of Hamilton's characteristic function  $S$

$$S = \int p_\phi d\phi + \int p_\theta d\theta + \int p_r dr \quad (23)$$

followed by use of the equations  $w_i = \partial S / \partial J_i$  (see [14] for the equivalent calculation for the Keplerian potential). We find that

$$w_1 = \frac{1}{4\pi} \arccos \left( \frac{A^2 \cos 2\phi - k_1 + k_2}{\sqrt{(A^2 - k_1 + k_2)^2 - 4A^2 k_2}} \right) - \frac{1}{4\pi} \arccos \left( \frac{2A^2 \cot^2 \theta - B^2 + A^2 + k_2}{\sqrt{(B^2 - A^2 - k_2)^2 - 4A^2 k_2}} \right) \quad (24)$$

$$w_2 = \frac{1}{4\pi} \arcsin \left( \frac{B^2 \cos 2\theta + A^2 - k_3}{\sqrt{(A^2 + B^2 - k_3)^2 - 4A^2 B^2}} \right) - \frac{1}{2\pi} \arcsin \left( \frac{kr - 2B^2}{r\sqrt{k^2 + 4B^2 E}} \right) \quad (25)$$

where

$$A = \frac{J_1}{2\sqrt{2}\pi} + \sqrt{k_1} + \sqrt{k_2} = \sqrt{I_2} \quad (26)$$

$$B = \frac{J_2}{2\sqrt{2}\pi} + \sqrt{k_1} + \sqrt{k_2} + \sqrt{k_3} = \sqrt{I_1} \quad (27)$$

Note that in the case that  $k_1 = k_2 = k_3 = 0$  (the Kepler problem), it is usual to introduce the inclination  $i = \arccos(A/B)$  of the orbital plane. Then, the angle  $w_1$  reduces to

$$w_1 = \frac{1}{2\pi} (\phi - \arcsin(\cot \theta \cot i)) \quad (28)$$

which is the longitude of the ascending node. The second angle  $w_2$  is easiest evaluated in the orbital plane with polar coordinates  $(r, \varphi)$  and becomes

$$w_2 = \frac{1}{2\pi} ((\varphi - \varphi_{\text{lan}}) - (\varphi - \varphi_{\text{peri}})) = \omega. \quad (29)$$

The first integral is therefore the angular difference between the orbital position and the longitude of the ascending node, the second the angular difference between the orbital position and the periapse. Thus,  $w_2$  reduces to  $\omega$ , the longitude of the periapse in the Kepler problem.

## V. SUMMARY AND CONCLUSIONS

We have found a new superintegrable Hamiltonian, which is a generalization of the well-known Kepler problem. There are five isolating integrals of the motion, namely the energy and generalizations of the components of the angular momentum and Laplace-Runge-Lenz vectors. Intriguingly, some of the integrals of motion are quartic in the momenta and do not arise from separability of the Hamilton-Jacobi equation.

There are three interesting questions which merit further research. First, it is clear that the  $N$  degrees of freedom Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{k}{r} + \frac{k_i}{x_i^2} \quad (30)$$

is also superintegrable. It would be interesting to find the complete set of  $2N-1$  functionally independent integrals of motion in this case. Second, it is well-known [15, 16] that the additional integrals of motion in the Kepler problem arise from the existence of the dynamical symmetry group  $SO(4)$ . It would be interesting to understand the group theoretic interpretation of the integrals of motion discussed in this paper. Third, although the reduction technique we used to generate the superintegrable potential in Section II is simple, it can be made to do some more work. For example, it is also clear that the Hamiltonian [c.f., eq (3)]

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \ell^2 x^2 + m^2 y^2 + n^2 z^2 + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}.$$

always has 5 independent integrals of motion as well, whose form remains to be established. Perhaps all superintegrable potentials in three degrees of freedom can be viewed as projections of higher dimensional Keplerian or harmonic oscillator motion?

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